# Jacobians and Determinants

Assume a simple linear system:

$$\dot{x} = \begin{bmatrix} -4 & -3 \\ 1.5 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0.2 \end{bmatrix} F \sin(\omega t),\ T = 0.01 \tag{1}$$

The solution of the system is:

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} BF \sin(\omega\tau) d\tau \tag{2}$$

The system will exhibit a limit cycle (since its eigenvalues are stable):

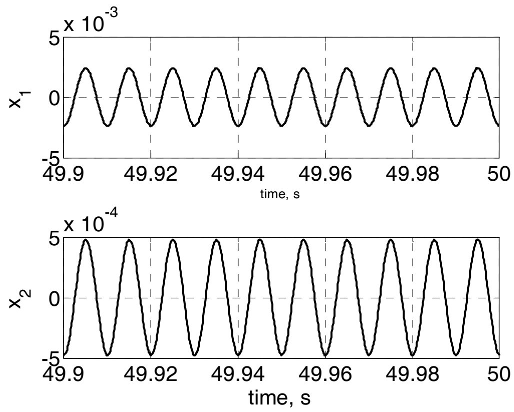


**Figure 1 Time responses**

Hence we can study this system using a Poincare map:

$$x(T) = e^{AT} x_0 + \int_0^T e^{A(T-\tau)} BF \sin(\omega\tau) d\tau \Leftrightarrow$$

$$x(k+1) = e^{AT} x(k) + \int_0^{kT} e^{A(kT-\tau)} BF \sin(\omega\tau) d\tau \tag{3}$$

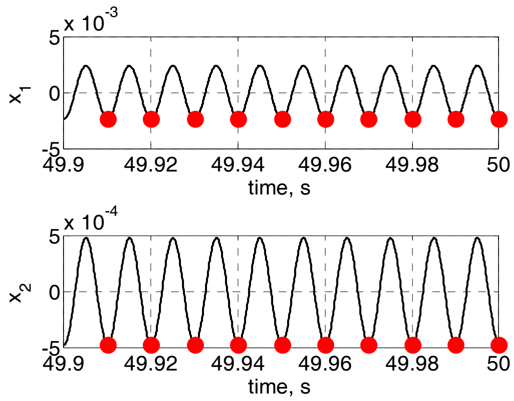


**Figure 2 Time responses and Poincare map points**

The fixed point of the map is:

$$x_{FP} = e^{AT} x_{FP} + \int_0^T e^{A(T-\tau)} BF \sin(\omega\tau) d\tau \Leftrightarrow$$
$$x_{FP} = \left(I - e^{AT}\right)^{-1} \int_0^T e^{A(T-\tau)} BF \sin(\omega\tau) d\tau \qquad (4)$$

Which can be found numerically by brute force (since the map is stable) or analytically:

**Box 1**

```
A=[-4 -3;1.5 1]; B=[1;0.2]; F=1.4974; T=0.01; w=2*pi/T; syms tau
Xep=inv(eye(2)-expm(A*T))*eval(int(expm(A*(T-tau))*F*sin(w*tau),0,T)*B);

Xep =
  -0.00238310583162
  -0.00047666861368

>> eig(A)

  ans =

    -2.82287565553230
    -0.17712434446770
```

To check that the answer is correct I use that as an IC:

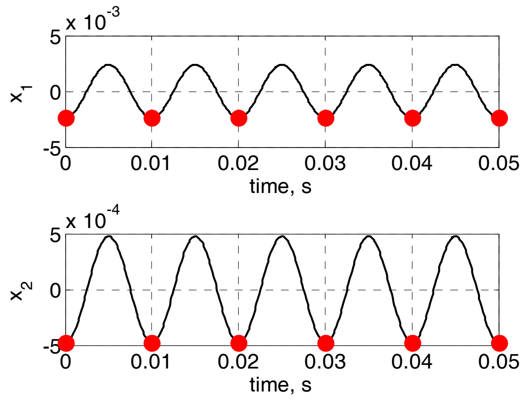


**Figure 3 Time responses and PM points starting from the FP**

Of course the LC would still be there even if A was unstable but then we would not be able to see it if we did not start on the LC. But using the previous method shown in Box 1:

**Box 2**

```
>> A=[-4 3;1.5 1]; B=[1;0.2]; F=1.4974; T=0.01; w=2*pi/T; syms tau
Xep=inv(eye(2)-expm(A*T))*eval(int(expm(A*(T-tau))*F*sin(w*tau),0,T)*B);
>> Xep

Xep =

  -0.00238307323825
  -0.00047665774671
>> eig(A)

  ans =

    -4.77871926215100
     1.77871926215100
```

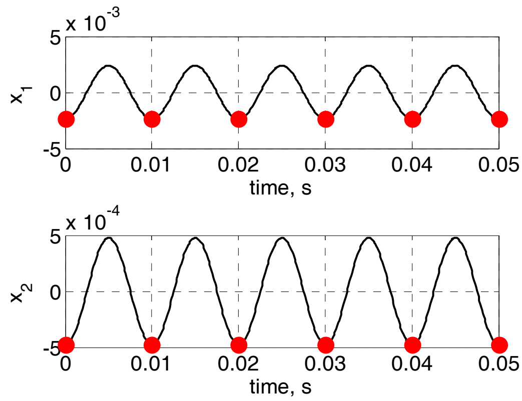


**Figure 4**

The Jacobian of that map is:

$$\frac{\partial x(k+1)}{\partial x(k)} = e^{AT} \tag{5}$$

Thus when we have a stable system the eigenvalues of A must be in the LHS and this implies that the eigenvalues of the expm(A*T) must be in the unit circle:

**Box 3**

```
>> A=[-4 -3;1.5 1];
>> eig(A)

ans =

  -2.82287565553230
  -0.17712434446770

>> eig(expm(A*T))

ans =

   0.97216595202841
   0.99823032428125

>> A=[-4 3;1.5 1];
eig(A)
eig(expm(A*T))

ans =

  -4.77871926215100
   1.77871926215100

ans =

   0.95333664256389
   1.01794632684904
```

But let’s probe (5) a little bit more:

$$\frac{\partial x(k+1)}{\partial x(k)} = e^{AT} \Leftrightarrow \partial x(k+1) = e^{AT} \partial x(k) \Leftrightarrow$$
$$x(k+1) - X_{ep} = e^{AT}\left(x(k) - X_{ep}\right) \tag{6}$$

With $x(k)$ being very close to $X_{ep}$ (otherwise we need nonlinear terms)

For example

**Box 4**

```
>> XA=[Xep(1)*(1+0.0001*(rand-0.5)) Xep(2)*(1+0.0001*(rand-0.5))]

XA =

  -0.00238320635412  -0.00047667996827
```

Then we expect that:

**Box 5**

```
expm(A*T)*(XA-Xep)'

ans =
   1.0e-007 *
  -0.96223278896487
  -0.12951601393510
```

The last point must be XA1-Xep. Thus running the Simulink model for T seconds starting from XA must get me to XA1:

**Box 6**

```
clc, clear, warning off; A=[-4 -3;1.5 1]; B=[1;0.2];
F=1.4974; T=0.01; w=2*pi/T;

syms tau
Xep=inv(eye(2)-expm(A*T))*eval(int(expm(A*(T-tau))*F*sin(w*tau),0,T)*B);

Xep=Xep'; Tstop=T;
XA=[Xep(1)*(1+0.0001*(rand-0.5)) Xep(2)*(1+0.0001*(rand-0.5))];  ; X0=XA;
sim('cont_disc2');
f=f(end,2:3); XA1=f;

XA1 =
  -0.00238320205490  -0.00047668156528
  >> XA1-Xep
  ans =
    1.0e-007 *
    -0.96223278886910  -0.12951601396883
```

The last one can be seen from a different point of view. If we start from 2 ICs close to the FP (say XA and XB) and we run the simulation for T seconds we get to 2 other points (XA1 and XB1). Then by assuming that $J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and using (6) we calculate the unknown numbers *a*, *b*, *c* & *d*:

**Box 7**

```
clc; clear, warning off; A=[-4 -3;1.5 1]; B=[1;0.2]; F=1.4974; T=0.01; w=2*pi/T;

syms tau; Xep=inv(eye(2)-expm(A*T))*eval(int(expm(A*(T-
tau))*F*sin(w*tau),0,T)*B);

Xep=Xep'; Tstop=T;
XA=[Xep(1)*(1+0.0001*(rand-0.5)) Xep(2)*(1+0.0001*(rand-0.5))];
X0=XA;
sim('cont_disc2'); f=f(end,2:3); XA1=f

XB=[Xep(1)*(1+0.0001*(rand-0.5)) Xep(2)*(1+0.0001*(rand-0.5))];
X0=XB;
sim('cont_disc2'); f=f(end,2:3); XB1=f;

syms a b c d; J=[a b;c d]

% expm(A*T)*(XA-Xep)'-(XA1-Xep)',  expm(A*T)*(XB-Xep)'-(XB1-Xep)'

F1=(XA1-Xep)'-J*(XA-Xep)';
F2=(XB1-Xep)'-J*(XB-Xep)';

h=solve(F1(1), F2(1)); g=solve(F1(2), F2(2));

Jeval=eval([h.a h.b;g.c g.d]);

Jeval =

   0.96056962154875  -0.02955422169800
   0.01477711001213   1.00982665548479

expm(A*T)

ans =
   0.96056962134768  -0.02955422016858
   0.01477711008429   1.00982665496198
```

### Transversal Intersections

The buck converter exhibits a transversal intersection:

**Box 9**

```
Uref=12; Vin=24;
L=20/1000; C=47/1000000; R=22;
TC=C*R; TL=L/R; T=1/2500;
A1=8.4; Ul=3.8; Uu=8.2;
```

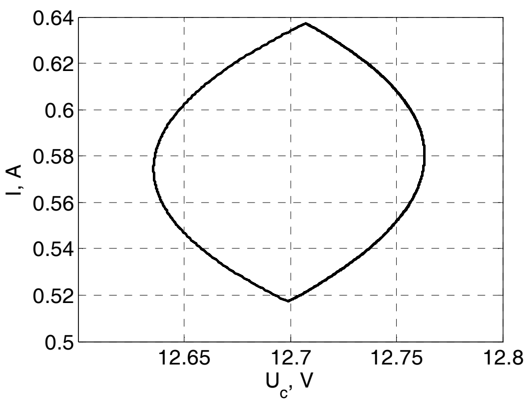


**Figure 5**

So can we apply the same method here?

In order to find the FP:

$$\mathbf{x}(dT) = \mathbf{\Phi}_1(dT)\mathbf{x}(0) \tag{7}$$

$$\mathbf{x}(T) = \mathbf{x}(0) = \mathbf{\Phi}_2(T)\mathbf{x}(dT) + \int_{dT}^{T} e^{\mathbf{A}_s(T-\tau)} \begin{bmatrix} 0 \\ \frac{V_{in}}{L} \end{bmatrix} d\tau \tag{8}$$

Hence: $\mathbf{x}(0) = \mathbf{\Phi}_2(T)\mathbf{\Phi}_1(dT)\mathbf{x}(0) + \int_{dT}^{T} e^{\mathbf{A}_s(T-\tau)} \begin{bmatrix} 0 \\ \frac{V_{in}}{L} \end{bmatrix} d\tau$

Therefore, $\mathbf{x}(0) = e^{\mathbf{A}_s T}\mathbf{x}(0) + \int_{dT}^{T} e^{\mathbf{A}_s(T-\tau)} \begin{bmatrix} 0 \\ \frac{Vin}{L} \end{bmatrix} d\tau \Leftrightarrow$

$$\Leftrightarrow \mathbf{x}(0) = \left[\mathbf{I} - e^{\mathbf{A}_s T}\right]^{-1} \int_{dT}^{T} e^{\mathbf{A}_s(T-\tau)} \begin{bmatrix} 0 \\ \frac{Vin}{L} \end{bmatrix} d\tau$$

From the hypersurface:

$$x_1(t_\Sigma) = U_{ref} + \frac{V_L + (V_U - V_L)d}{A} \tag{9}$$

But we know that $x_1(t_\Sigma) = [1 \quad 0] e^{\mathbf{A}_s dT}\mathbf{x}(0)$ (10)

Hence, $f(d) = [1 \quad 0] e^{\mathbf{A}_s dT}\mathbf{x}(0) - U_{ref} - \frac{V_L + (V_U - V_L)d}{A} = 0$ (11)

$$f(d) = [1 \quad 0] e^{\mathbf{A_s} dT} \left( \left[ \mathbf{I} - e^{\mathbf{A_s} T} \right]^{-1} \int_{dT}^{T} e^{\mathbf{A_s}(T-\tau)} \begin{bmatrix} 0 \\ \frac{V_{in}}{L} \end{bmatrix} d\tau \right) - U_{ref} - \frac{V_L + (V_U - V_L) d}{A} = 0 \quad (12)$$

The mfile for that is:

**Box 10**

```
clc; clear; cnt=1; syms d, tau

Uref=12; Vin=24; L=20/1000; C=47/1000000; R=22; TC=C*R; TL=L/R; T=1/2500;
A1=8.4; Ul=3.8; Uu=8.2; A_s=[-1/R/C 1/C; -1/L 0];

xd0=inv(eye(2)-expm(A_s*T))*int(expm(A_s*(T-tau))*[0;Vin/L],tau,d*T,T);
x1dT=[1 0]*expm(A_s*d*T)*xd0;

f=A1*([1 0]*expm(A_s*d*T)*xd0-Uref)-Ul-(Uu-Ul)*d;

fd=diff(f,d);
x=0.99;
for k=1:10
   y(k)=x;
   x=x-subs(f,d,x)/subs(fd,d,x);
end

d=x;
X0=eval(inv(eye(2)-expm(A_s*T))*int(expm(A_s*(T-tau))*[0;Vin/L],tau,d*T,T))
```

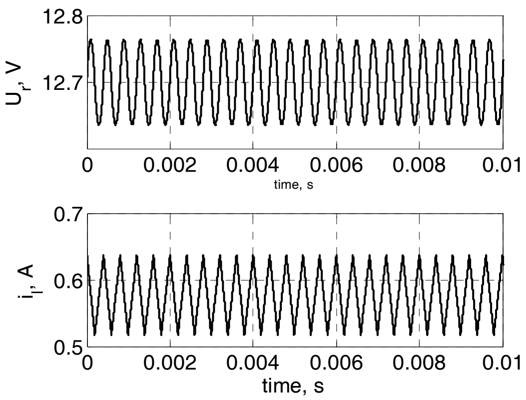


**Figure 6**

The same analysis applies but now we have to make sure that the perturbations are **very small**. The mfile "newt_rap_int.m" does the above calculation. The results are: **Everything is extremely sensitive to numerical errors**.

**Box 11**

```
Y =
  -0.91635976412080   0.03747635529883
  -0.38656602453153  -0.72537862471831
Jeval =
  -0.91639876796125   0.03638740747461
  -0.38657251814250  -0.72555951429268
```